

## SUPERCOHERENT STATES AND PHYSICAL SYSTEMS

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A method is developed for obtaining coherent states of a system admitting a supersymmetry. These states are called supercoherent states. The approach presented in this talk is based on an extension to supergroups of the usual group-theoretic approach. The example of the supersymmetric harmonic oscillator is discussed, thereby illustrating some of the attractive features of the method. Supercoherent states of an electron moving in a constant magnetic field are also described.

**1. Introduction**

Over the past three decades, the notion of coherent state [1-6] has enjoyed a significant role in diverse areas of physics. Several basic definitions are in use [7]. For example, among the possibilities for the simple harmonic oscillator are the definition as eigenstates of the annihilation operator, the one as states having and preserving minimum uncertainty, and the one via the displacement operator. All these yield the same harmonic-oscillator coherent states, representing a gaussian wavepacket preserving its shape while executing the classical motion.

This talk describes a generalization of the concept of coherent states to that of supercoherent states, relevant for systems admitting one or more supersymmetries. A supersymmetry involves both bosonic and fermionic states, and the

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corresponding symmetry generators close under a combination of commutation and anticommutation relations into a superalgebra. The additional structure this entails means that the physically appropriate generalization of coherent states to supercoherent states is not immediately apparent.

Our solution to this problem involves an extension to supergroups of a generalized method [6] for ordinary coherent states that is based on Lie groups and involves use of the Baker-Campbell-Hausdorff (BCH) relations [8-13] connecting different group parametrizations. Supergroups can be viewed as extensions of Lie groups with Grassmann-valued parameters. The theory of supergroups considered both as abstract groups and as superanalytic supermanifolds has been developed [14-16], and methods for obtaining BCH relations for supergroups are known [17-19]. A summary of our methods is provided in section 2.

As an example of the method, the supercoherent states for the supersymmetric harmonic oscillator are considered in section 3. The supersymmetry for this case is generated by the super Heisenberg-Weyl algebra, containing the identity and bosonic and fermionic creation and annihilation operators. It is closely related to supersymmetric quantum mechanics [20-29], which is applicable in several physical situations. An example with relevance to the quantum Hall effect is the case of an electron moving in a constant magnetic field [28,29]. This situation is considered in section 4.

The reader is referred to [30], on which this talk is based, for more information about our general construction of supercoherent states, about its relation to other approaches [31-33], and about applications in various physical situations.

## 2. Method

There is a close connection between group theory and coherent states. To see this for the simple harmonic oscillator, consider the usual approach via the displacement operator  $D$ , given by  $D(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a)$ . This displaces the annihilation operator  $a$  by a complex constant  $\alpha$ :  $D^{-1}(\alpha) a D(\alpha) = a + \alpha$ . The operator  $D$  is a unitary element of the harmonic-oscillator symmetry group, called the Heisenberg-Weyl group, for which the associated algebra is  $[a, a^\dagger] = 1$ . By definition, the coherent state parametrized by  $\alpha$  is given by the action of  $D(\alpha)$  on the ground state  $|0\rangle$ . The correct normalization of  $|\alpha\rangle$  is fixed by the unitarity of  $D$ . The form of  $|\alpha\rangle$  can then be explicitly exhibited using the BCH relation  $e^A e^B = e^{(A+B+\frac{1}{2}[A,B])}$ , valid for any two operators  $A$  and  $B$  both commuting with  $[A, B]$ .

For a general system with an arbitrary Lie group  $G$  as symmetry group, coherent states can be defined as follows [3,6]. Given a unitary irreducible representation  $T(g)$  of  $G$  acting in a Hilbert space  $H$ , set  $|\Psi_0\rangle$  as some given element

in  $H$ . The coherent states are then the set  $\{|\Psi_g\rangle\} = \{T(g)|\Psi_0\rangle\}$ . This definition is parallel to the displacement-operator approach for the harmonic oscillator.

For systems admitting supersymmetry, we extend this method to supergroups using the construction of refs. [14-16]. In this approach, supergroups are defined in analogy with the definition of Lie groups via analytic manifolds, using Grassmann-valued parameters instead of real or complex ones. The resulting supergroup coordinates include both commuting and anticommuting variables. We refer the reader to refs. [14-16] for details of the construction. A summary of the essential points is contained in the paper [30] on which this talk is based.

To find supercoherent states via the group-theoretic method requires the use of unitary supergroup representations. Introduce the supergroup generators  $B_j, F_\alpha$ , where the corresponding superalgebra\* involves commutators among the  $B_j$  and anticommutators among the  $F_\alpha$ . Choose a superhermitian basis [31], i.e., set  $B_j^\dagger = B_j$  and  $F_\alpha^\dagger = -F_\alpha$ . Then, a general unitary supergroup element is  $T(g) = \exp(A_j B_j + \theta_\alpha F_\alpha)$ , where  $A_j$  is real Grassmann commuting and  $\theta_\alpha$  is real Grassmann anticommuting.

Supercoherent states are found by applying  $T(g)$  to an extremal state in the (super) Hilbert space. To find explicit expressions requires the use of BCH relations for the supergroup. A general method for determining these and specific formulae for some frequently used supergroups may be found in refs. [17-19].

### 3. The Supersymmetric Harmonic Oscillator

By definition, the hamiltonian  $H$  of a supersymmetric quantum-mechanical system [20-23] commutes with  $N$  supersymmetry operators  $Q_j$  of which it is a quadratic function:  $\delta_{jk}H = \{Q_j, Q_k\}$ . The superalgebra generated by  $H$  and  $Q_j$  is called  $sqm(N)$ . Choosing  $N = 2$  gives  $sqm(2)$ , which appears in several physical contexts [24-29]. Defining  $Q = (Q_1 + iQ_2)/\sqrt{2}$  and  $Q^\dagger = (Q_1 - iQ_2)/\sqrt{2}$ , the superalgebra  $sqm(2)$  is  $H = \{Q, Q^\dagger\}$ ,  $[H, Q] = [H, Q^\dagger] = 0$ .

The supersymmetric quantum harmonic oscillator can be defined in terms of annihilation and creation operators  $a, a^\dagger; b, b^\dagger$  generating a supersymmetric extension of the usual Heisenberg-Weyl algebra:  $[a, a^\dagger] = \{b, b^\dagger\} = \mathbf{I}$ . The corresponding super Hilbert space is spanned by states  $|n, \nu\rangle$ , where  $n = 0, 1, 2, \dots$  and  $\nu = 0, 1$ . States with  $\nu = 0$  are called bosonic and those with  $\nu = 1$  are called fermionic.

The  $sqm(2)$  superalgebra is generated by the oscillator hamiltonian  $H = a^\dagger a + b^\dagger b$  and by the supersymmetry operators  $Q = ab^\dagger, Q^\dagger = a^\dagger b$ . It follows from

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\* For an overview of superalgebras, see ref. [34]

$H|n, \nu\rangle = (n + \nu)|n, \nu\rangle$  that  $|n, 0\rangle$  and  $|n - 1, 1\rangle$  are degenerate states for all  $n$  except  $n = 0$ . The ground state  $|0, 0\rangle$  is thus unique. Unbroken supersymmetry,  $Q|0, 0\rangle = Q^\dagger|0, 0\rangle = 0$ , implies that the ground state has energy eigenvalue zero. The generator  $Q^\dagger$  takes bosonic states into fermionic ones, while  $Q$  takes fermionic states into bosonic ones.

Following the method described in section 2, supercoherent states for the supersymmetric oscillator are given in terms of a unitary representation  $T(g)$  of the super Heisenberg-Weyl group. The supergroup element of relevance may be taken as  $T(g) = \exp(-\bar{A}a + Aa^\dagger + \theta b^\dagger + \bar{\theta}b)$  where  $A$  is complex Grassmann commuting and  $\theta$  is complex Grassmann anticommuting. The necessary BCH relation for the super Heisenberg-Weyl group, needed for explicit calculation of the supercoherent states, is found using Lemma 1 of ref. [17]. The result is

$$T(g) = \exp\left(\frac{1}{2}\theta\bar{\theta} - \frac{1}{2}|A|^2\right)\exp(Aa^\dagger)\exp(\theta b^\dagger)\exp(-\bar{A}a)\exp(\bar{\theta}b) \quad (3.1)$$

The supercoherent states  $|Z\rangle$  are obtained by applying  $T(g)$  to the ground state  $|0, 0\rangle$ . They are given by

$$|Z\rangle = \left(1 + \frac{1}{2}\theta\bar{\theta}\right)|A, 0\rangle + \theta|A, 1\rangle \quad (3.2)$$

where for convenience we have defined  $|A, \nu\rangle = \exp(-|A|^2/2)\exp(Aa^\dagger)|0, \nu\rangle$ .

The supercoherent states  $|Z\rangle$  have the following attractive properties, all of which are natural generalizations of the corresponding features of ordinary harmonic-oscillator coherent states.

- They are defined via a natural extension of the usual displacement operator approach.
- They are eigenstates of the annihilation operators  $a$  and  $b$ :  $a|Z\rangle = A|Z\rangle$ ,  $b|Z\rangle = -\theta|Z\rangle$ .
- They maintain the minimum-uncertainty value  $\Delta q \Delta p = \frac{1}{2}$  in time.
- They are unity normalized,  $\langle Z|Z\rangle = 1$ .
- They are not orthogonal and form an (over)complete set. The identity is resolved by  $\int |Z\rangle\langle Z| d\bar{\theta}d\theta dA = \pi \mathbf{I}$ .
- They yield the usual harmonic-oscillator coherent states  $|A\rangle$  when  $\theta = 0$ .
- They contain as the subset  $A = 0$  the usual fermionic coherent states [35] for a single anticommuting fermionic degree of freedom.

#### 4. A Physical Example

The quantum system consisting of a nonrelativistic electron of mass  $M$  and charge  $e$  moving in a constant uniform magnetic field  $\mathbf{B} = B\hat{z}$  provides a physical

realization of supersymmetric quantum mechanics [28,29]. The wavefunctions  $e^{-iEt}\psi(\vec{r})$  for this system obey the two-component Pauli equation, which reduces to  $H\psi = E\psi$  with  $H \equiv \frac{1}{2M}[\vec{\sigma} \cdot (\vec{p} - e\vec{A})]^2$ . The use of cylindrical coordinates is natural, as is the choice of cylindrical gauge  $A_x = -\frac{1}{2}By$ ,  $A_y = \frac{1}{2}Bx$ . For simplicity, we restrict the analysis to the two-dimensional problem, so that  $p_z = 0$ .

The explicit realization of the super Heisenberg-Weyl algebra is as follows. Define the dimensionless quantities  $\hat{H} = MH/eB$ ,  $\hat{E} = ME/eB$ , and introduce the annihilation operators

$$a = \frac{1}{\sqrt{2eB}} e^{i\varphi} \left( \partial_r + \frac{i}{r} \partial_\varphi + \frac{1}{2} eBr \right) \quad (4.1)$$

and

$$b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} . \quad (4.2)$$

Then, the Pauli equation takes the manifestly supersymmetric form

$$\hat{H}\psi \equiv (a^\dagger a + b^\dagger b)\psi = \hat{E}\psi . \quad (4.3)$$

All the features of the supersymmetric harmonic oscillator discussed in section 3 are reproduced. Note that the fermion annihilation operator  $b$  acts to reverse the electron spin, and therefore the  $sqm(2)$  generator  $Q$  does also.

Equation (4.3) is equivalent to a confluent hypergeometric equation with two-component solutions labeled by two quantum numbers, one related to the energy eigenvalue  $\hat{E}$  and one labeling degenerate eigenstates. The explicit solution is given in our paper [30]. We write  $\psi = |n, m; \nu\rangle$ , where the upper and lower components of  $\psi$  are labeled by  $\nu = 0$  and  $\nu = 1$ , respectively. The operators  $a$  and  $a^\dagger$  act as canonical lowering and raising operators on the quantum number  $n$ , while  $b$  and  $b^\dagger$  act on  $\nu$ . To form a complete set, introduce

$$c^\dagger = -\frac{1}{\sqrt{2eB}} e^{i\varphi} \left( \partial_r + \frac{i}{r} \partial_\varphi - \frac{1}{2} eBr \right) , \quad (4.4)$$

acting as a canonical lowering operator on  $m$  and satisfying  $[c, c^\dagger] = 1$ . The full supergroup for this physical system is therefore the product of the super Heisenberg-Weyl group (generated by  $a$ ,  $b$ , and conjugates) with another Heisenberg-Weyl group (generated by  $c$  and conjugate).

The supercoherent states can now be constructed via the method of section 2. Their explicit form is quickly found from eq. (3.2) by noting that coherent states with respect to  $c$  and  $c^\dagger$  are the usual harmonic-oscillator coherent states and that  $c$  and  $c^\dagger$  commute with all other generators. The result is

$$|Z\rangle = \exp\left(\frac{1}{2}\theta\bar{\theta}\right) \exp\left(-\frac{|A|^2}{2}\right) \exp\left(-\frac{|C|^2}{2}\right) \sum_{n,m} \frac{A^n C^m}{\sqrt{n!}\sqrt{m!}} \left( |n, m; 0\rangle + \theta |n, m; 1\rangle \right) . \quad (4.5)$$

These supercoherent states depend on three Grassmann-valued variables,  $A$ ,  $C$ , and  $\theta$ . It can be shown that all the attractive features of the oscillator supercoherent states discussed in section 3 are reproduced.

The expectation values of the hamiltonian  $H$ ,  $\langle Z|H|Z\rangle = \frac{eB}{M}(A\bar{A} - \theta\bar{\theta})$ , and of the magnetic-moment interaction energy  $U = -eB\sigma_z/2M$ ,  $\langle Z|U|Z\rangle = -\frac{eB}{2M}(1 + 2\theta\bar{\theta})$ , provide insight into the role of the Grassmann-valued variables in Eq. (4.5). The difference  $\langle Z|H - U|Z\rangle = \frac{eB}{M}(A\bar{A} + \frac{1}{2})$  represents the energy expectation in the absence of the magnetic moment. It is independent of  $\theta\bar{\theta}$  and the value of  $A\bar{A}$  is shifted by one half. Since the magnetic moment  $U$  distinguishes between eigenstates with  $\nu = 0$  and  $\nu = 1$ , it follows that the term with  $\theta\bar{\theta}$  contains the information about the energy splitting between the two sets of eigenstates.

As we have seen, the supersymmetry present in this physical system ensures a group-theoretical and natural incorporation of the electron spin. This feature of supersymmetry is manifest in other physical systems. For instance, one key aspect of atomic and ionic supersymmetry [25] is the natural appearance of the Pauli principle.

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